3.2: Complements, Intersections, and Unions

Learning Objectives

- To learn how some events are naturally expressible in terms of other events.
- To learn how to use special formulas for the probability of an event that is expressed in terms of one or more other events.

Some events can be naturally expressed in terms of other, sometimes simpler, events.

Complements

Definition: Complement

The complement of an event A in a sample space S, denoted A^c, is the collection of all outcomes in S that are not elements of the set A. It corresponds to negating any description in words of the event A.

Example $\PageIndex{1}$

Two events connected with the experiment of rolling a single die are E**:** ***"the number rolled is even"*** and T**:** ***"the number rolled is greater than two."*** Find the complement of each.

Solution

In the sample space $S=\{1,2,3,4,5,6\}$ the corresponding sets of outcomes are $E=\{2,4,6\}$ and $T=\{3,4,5,6\}$. The

complements are $E^c=\{1,3,5\}$ and $T^c=\{1,2\}$.

In words the complements are described by “the number rolled is not even” and “the number rolled is not greater than two.” Of course easier descriptions would be “the number rolled is odd” and “the number rolled is less than three.”

If there is a 60% chance of rain tomorrow, what is the probability of fair weather? The obvious answer, 40%, is an instance of the following general rule.

Definition: Probability Rule for Complements

The Probability Rule for Complements states that $$P(A^c) = 1 - P(A) \nonumber$$

This formula is particularly useful when finding the probability of an event directly is difficult.

Example $\PageIndex{2}$

Find the probability that at least one heads will appear in five tosses of a fair coin.

Solution

Identify outcomes by lists of five hs and ts, such as $tthtt$ and $hhttt$. Although it is tedious to list them all, it is not difficult to count them. Think of using a tree diagram to do so. There are two choices for the first toss. For each of these there are two choices for the second toss, hence $2\times 2 = 4$ outcomes for two tosses. For each of these four outcomes, there are two possibilities for the third toss, hence $4\times 2 = 8$ outcomes for three tosses. Similarly, there are $8\times 2 = 16$ outcomes for four tosses and finally $16\times 2 = 32$ outcomes for five tosses.

Let O denote the event “at least one heads.” There are many ways to obtain at least one heads, but only one way to fail to do so: all tails. Thus although it is difficult to list all the outcomes that form O, it is easy to write $O^c = \{ttttt\}$. Since there are 32 equally likely outcomes, each has probability $\frac{1}{32}$, so $P(O^c)=1/32$, hence $P(O) = 1-\frac{1}{32}\approx 0.97$ or about a 97% chance.

Intersection of Events

Definition: intersections

The intersection of events A and B, denoted $A\cap B$, is the collection of all outcomes that are elements of both of the sets A and B. It corresponds to combining descriptions of the two events using the word “and.”

To say that the event $A\cap B$ occurred means that on a particular trial of the experiment both A and B occurred. A visual representation of the intersection of events A and B in a sample space S is given in Figure $\PageIndex{1}$. The intersection corresponds to the shaded lens-shaped region that lies within both ovals.

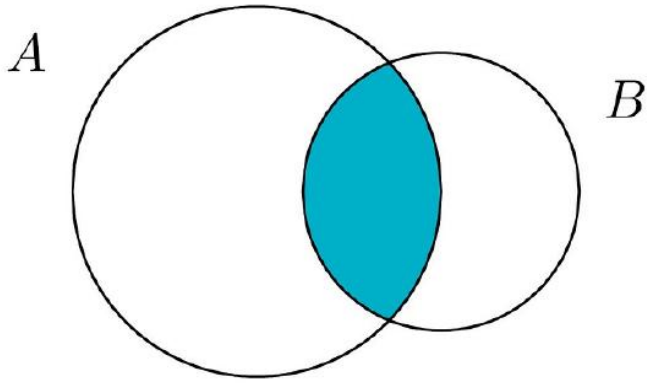

Figure \(\PageIndex{1}\)*: The Intersection of Events A and B*

Example \(\PageIndex{3}\)

In the experiment of rolling a single die, find the intersection \(E\cap T\) of the events \(E\): *"the number rolled is even"* and \(T\): *"the number rolled is greater than two."*

Solution

The sample space is \(S=\{1,2,3,4,5,6\}\). Since the outcomes that are common to \(E=\{2,4,6\}\) and \(T=\{3,4,5,6\}\) are \(4\) and \(6\), \(E\cap T=\{4,6\}\).

In words the intersection is described by "the number rolled is even and is greater than two." The only numbers between one and six that are both even and greater than two are four and six, corresponding to \(E\cap T\) given above.

Example \(\PageIndex{4}\)

A single die is rolled.

1. Suppose the die is fair. Find the probability that the number rolled is both even and greater than two.
2. Suppose the die has been "loaded" so that \(P(1)=\frac{1}{12}\), \(P(6)=\frac{3}{12}\), and the remaining four outcomes are equally likely with one another. Now find the probability that the number rolled is both even and greater than two.

Solution

In both cases the sample space is \(S=\{1,2,3,4,5,6\}\) and the event in question is the intersection \(E\cap T=\{4,6\}\) of the previous example.

1. Since the die is fair, all outcomes are equally likely, so by counting we have \(P(E\cap T)=\frac{2}{6}\).
2. The information on the probabilities of the six outcomes that we have so far is

\[\begin{array}{l|cccc}Outcome & 1 & 2 & 3 & 4 & 5 & 6 \\ Probablity & \frac{1}{12} & p & p & p & p & \frac{3}{12}\end{array} \nonumber \]

Since $P(1)+P(6)=\frac{4}{6}=\frac{1}{3}$

$$P(2) + P(3) + P(4) + P(5) = 1 - \frac{1}{3} = \frac{2}{3}$$

Thus $4p=\frac{2}{3}$, so $p=\frac{1}{6}$. In particular $P(4)=\frac{1}{6}$ therefore:

$$P(E\cap T) = P(4) + P(6) = \frac{1}{6} + \frac{3}{12} = \frac{5}{12}$$

Definition: mutually exclusive

Events A and B are mutually exclusive (cannot both occur at once) if they have no elements in common.

For A and B to have no outcomes in common means precisely that it is impossible for both A and B to occur on a single trial of the random experiment. This gives the following rule:

Definition: Probability Rule for Mutually Exclusive Events

Events A and B are mutually exclusive if and only if

$$P(A \cap B) = 0$$

Any event A and its complement A^c are mutually exclusive, but A and B can be mutually exclusive without being complements.

Example $\PageIndex{5}$

In the experiment of rolling a single die, find three choices for an event A so that the events A and E: “the number rolled is even” are mutually exclusive.

Solution

Since $E=\{2,4,6\}$ and we want A to have no elements in common with E, any event that does not contain any even number will do. Three choices are $\{1,3,5\}$ (the complement E^c, the odds), $\{1,3\}$, and $\{5\}$.

Union of Events

Definition: Union of Events

The union of events A and $B,$ denoted $A\cup B$, is the collection of all outcomes that are elements of one or the other of the sets A and B, or of both of them. It corresponds to combining descriptions of the two events using the word “or.”

To say that the event $A\cup B$ occurred means that on a particular trial of the experiment either A or B occurred (or both did). A visual representation of the union of events A and B in a sample space S is given in Figure

\(\PageIndex{2}\). The union corresponds to the shaded region.

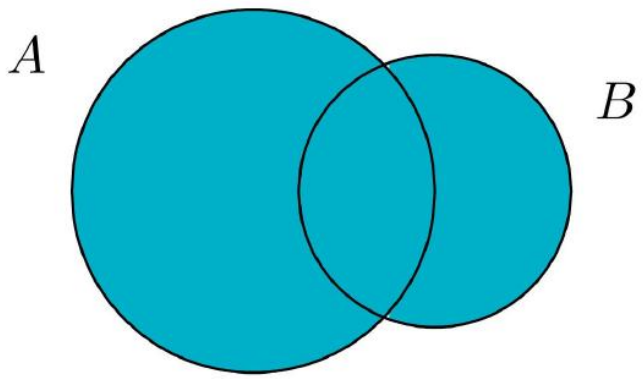

***Figure* \(\PageIndex{2}\)**: *The Union of Events A and B*

Example \(\PageIndex{6}\)

In the experiment of rolling a single die, find the union of the events \(E\)**: "the number rolled is even"** and \(T\)**: "the number rolled is greater than two."**

Solution

Since the outcomes that are in either \(E=\{2,4,6\}\) or \(T=\{3,4,5,6\}\) (or both) are \(2, 3, 4, 5,\) and \(6\), that means \(E\cup T=\{2,3,4,5,6\}\).

Note that an outcome such as \(4\) that is in both sets is still listed only once (although strictly speaking it is not incorrect to list it twice).

In words the union is described by "the number rolled is even or is greater than two." Every number between one and six except the number one is either even or is greater than two, corresponding to \(E\cup T\) given above.

Example \(\PageIndex{7}\)

A two-child family is selected at random. Let \(B\) denote the event that at least one child is a boy, let \(D\) denote the event that the genders of the two children differ, and let \(M\) denote the event that the genders of the two children match. Find \(B\cup D\) and \(B\cup M\).

Solution

A sample space for this experiment is \(S=\{bb,bg,gb,gg\}\), where the first letter denotes the gender of the firstborn child and the second letter denotes the gender of the second child. The events \(B, D,\) and \(M\) are \(B=\{bb,bg,gb\}\), \(D=\{bg,gb\}\), \(M=\{bb,gg\}\).

Each outcome in \(D\) is already in \(B\), so the outcomes that are in at least one or the other of the sets \(B\) and \(D\) is

just the set B itself: $B\cup D=\{bb,bg,gb\}=B$.

Every outcome in the whole sample space S is in at least one or the other of the sets B and M, so $B\cup M=\{bb,bg,gb,gg\}=S$.

Definition: Additive Rule of Probability

A useful property to know is the Additive Rule of Probability, which is

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) \nonumber$$

The next example, in which we compute the probability of a union both by counting and by using the formula, shows why the last term in the formula is needed.

Example $\PageIndex{8}$

Two fair dice are thrown. Find the probabilities of the following events:

1. both dice show a four
2. at least one die shows a four

Solution

As was the case with tossing two identical coins, actual experience dictates that for the sample space to have equally likely outcomes we should list outcomes as if we could distinguish the two dice. We could imagine that one of them is red and the other is green. Then any outcome can be labeled as a pair of numbers as in the following display, where the first number in the pair is the number of dots on the top face of the green die and the second number in the pair is the number of dots on the top face of the red die.

$$\begin{array}11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66\end{array} \nonumber$$

1. There are 36 equally likely outcomes, of which exactly one corresponds to two fours, so the probability of a pair of fours is $1/36$.
2. From the table we can see that there are 11 pairs that correspond to the event in question: the six pairs in the fourth row (the green die shows a four) plus the additional five pairs other than the pair 44, already counted, in the fourth column (the red die is four), so the answer is $11/36$. To see how the formula gives the same number, let A_G denote the event that the green die is a four and let A_R denote the event that the red die is a four. Then clearly by counting we get: $P(A_G) = 6/36$ and $P(A_R) = 6/36$. Since $A_G\cap A_R = \{44\}$, $P(A_G\cap A_R) = 1/36$. This is the computation from part 1, of course. Thus by the Additive Rule of Probability we get:

$$P(A_G\cap A_R) = P(A_G) + P(A_R) - P(A_G - A_R) = 6/36 + 6/36 - 1/36 = \frac{11}{36} \nonumber$$

Example $\PageIndex{9}$

A tutoring service specializes in preparing adults for high school equivalence tests. Among all the students seeking help from the service, 63% need help in mathematics, 34% need help in English, and 27% need help in both mathematics and English. What is the percentage of students who need help in either mathematics or English?

Solution

Imagine selecting a student at random, that is, in such a way that every student has the same chance of being selected. Let M denote the event "the student needs help in mathematics" and let E denote the event "the student needs help in English." The information given is that $P(M) = 0.63$, $P(E) = 0.34$ and $P(M\cap E) = 0.27$. Thus the Additive Rule of Probability gives:

$$P(M\cup E) = P(M) + P(E) - P(M\cap E) = 0.63 + 0.34 - 0.27 = 0.70 \nonumber$$

Note how the naïve reasoning that if 63% need help in mathematics and 34% need help in English then 63 plus 34 or 97% need help in one or the other gives a number that is too large. The percentage that need help in both subjects must be subtracted off, else the people needing help in both are counted twice, once for needing help in mathematics and once again for needing help in English. The simple sum of the probabilities would work if the events in question were mutually exclusive, for then $P(A\cap B)$ is zero, and makes no difference.

Example $\PageIndex{10}$

Volunteers for a disaster relief effort were classified according to both specialty (C: construction, E: education, M: medicine) and language ability (S: speaks a single language fluently, T: speaks two or more languages fluently). The results are shown in the following two-way classification table:

Specialty	**Language Ability**	
	S	**T**
C	12	1
E	4	3
M	6	2

The first row of numbers means that 12 volunteers whose specialty is construction speak a single language fluently, and 1 volunteer whose specialty is construction speaks at least two languages fluently. Similarly for the other two rows.

A volunteer is selected at random, meaning that each one has an equal chance of being chosen. Find the probability that:

1. his specialty is medicine and he speaks two or more languages;
2. either his specialty is medicine or he speaks two or more languages;

3. his specialty is something other than medicine.

Solution

When information is presented in a two-way classification table it is typically convenient to adjoin to the table the row and column totals, to produce a new table like this:

Specialty	Language Ability		Total
	\(S\)	\(T\)	
\(C\)	12	1	13
\(E\)	4	3	7
\(M\)	6	2	8
Total	22	6	28

1. The probability sought is \(P(M\cap T)\). The table shows that there are \(2\) such people, out of \(28\) in all, hence \(P(M\cap T) = 2/28 \approx 0.07\) or about a \(7\%\) chance.
2. The probability sought is \(P(M\cup T)\). The third row total and the grand total in the sample give \(P(M) = 8/28\). The second column total and the grand total give \(P(T) = 6/28\). Thus using the result from part (1),

   \[P(M\cup T) = P(M) + P(T) - P(M\cap T) = 828 + 628 - 228 = 1228\approx 0.43 \nonumber \]

   or about a \(43\%\) chance.

3. This probability can be computed in two ways. Since the event of interest can be viewed as the event \(C\cup E\) and the events \(C\) and \(E\) are mutually exclusive, the answer is, using the first two row totals,

   \[P(C\cup E) = P(C) + P(E) - P(C\cap E) = 1328 + 728 - 028 = 2028\approx 0.71 \nonumber \]

On the other hand, the event of interest can be thought of as the complement \(M^c\) of \(M\), hence using the value of \(P(M) \)computed in part (2),

\[P(M^c) = 1 - P(M) = 1 - 828 = 2028\approx 0.71 \nonumber \]

as before.

Key Takeaway

- The probability of an event that is a complement or union of events of known probability can be computed using formulas.