Simultaneous Inference

We want to find confidence interval for more than one parameter simultaneously. For example we might want to find confidence interval for \( \beta_0 \) and \( \beta_1 \).

Bonferroni Joint Confidence Intervals

The confidence coefficients for individual parameters are adjusted to the higher \( 1 - \alpha \) so that the confidence coefficient for the collection of parameters must be at least \( 1 - \alpha \). This is based on the following inequality:

Theorem (Bonferroni's Inequality)

\[
P(\beta_0 \cap \beta_1) \geq 1 - P(\beta_0^c) - P(\beta_1^c) \quad \text{(Equation \ref{Bonferroni})}
\]

for any two events \( \beta_0 \) and \( \beta_1 \), where \( \beta_0^c \) and \( \beta_1^c \) are complements of events \( \beta_0 \) and \( \beta_1 \), respectively.

We take, \( \beta_0 = \) the event that confidence interval for \( \beta_0 \) covers \( \beta_0 \); and, \( \beta_1 = \) the event that confidence interval for \( \beta_1 \) covers \( \beta_1 \);

So, if \( P(\beta_0) = 1 - \alpha_1 \), and \( P(\beta_1) = 1 - \alpha_2 \), then \( P(\beta_0 \cap \beta_1) \geq 1 - \alpha_1 - \alpha_2 \), by Bonferroni's inequality (Equation \ref{Bonferroni}). Note that \( \beta_0 \cap \beta_1 \) is the event that confidence intervals for both the parameters cover the respective parameters. Therefore we take \( \beta_1 = \alpha_1 = \alpha_2 = \alpha/2 \) to get joint confidence intervals with confidence coefficient at least \( 1 - \alpha \),

\[
(b_0 \pm t(1-\alpha/4;n-2) s(b_0)) \quad \text{and} \quad (b_1 \pm t(1-\alpha/4;n-2) s(b_1))
\]

for \( \beta_0 \) and \( \beta_1 \), respectively.
Bonferroni Joint Confidence Intervals for Mean Response

We want to find the simultaneous confidence interval for \(E(Y|X = X_h) = \beta_0 + \beta_1X_h\) for \(g\) different values of \(X_h\). Using *Bonferroni's inequality* for the intersection of \(g\) different events, the confidence intervals with confidence coefficient (at least) \((1-\alpha)\) are given by

\[
\hat{Y}_h \pm t(1-\alpha/2g; n-2)s(\hat{Y}_h).\]

Confidence band for regression line : Working-Hotelling procedure

The confidence band

\[
\hat{Y}_h \pm \sqrt{2F(1-\alpha;2,n-2)}s(\hat{Y}_h)
\]

contains the entire regression line (for all values of \(X\)) with confidence level \((1-\alpha)\). The Working-Hotelling procedure for obtaining the \((1-\alpha)\) simultaneous confidence band for the mean responses, therefore, is to use these confidence limits for the \(g\) different values of \(X_h\).

Simultaneous prediction intervals

Recall that, the standard error of prediction for a new observation \(Y_{h(new)}\) with \(X = X_h\), is

\[
\begin{align*}
\text{se}(Y_{h(new)}) &= \sqrt{MSE(1+\frac{1}{n}+\frac{(X_h-\overline{X})^2}{\sum_i(X_i-\overline{X})^2})} \\
\end{align*}
\]

In order to predict the new observations for \(g\) different values of \(X\), we may use one of the two procedures:

- **Bonferroni procedure** : \(\hat{Y}_h \pm t(1-\alpha/2g; n-2)s(Y_{h(new)}-\hat{Y}_h)\).
- **Scheffe procedure** : \(\hat{Y}_h \pm \sqrt{gF(1-\alpha;g,n-2)}s(Y_{h(new)}-\hat{Y}_h)\).

Remark : A point to note is that except for the Working-Hotelling procedure for finding simultaneous confidence intervals for mean response, in all the other cases, the confidence intervals become wider as \(g\) increases.

Which method to choose : Choose the method which leads to narrower intervals. As a comparison between Bonferroni and Working-Hotelling (for finding confidence intervals for the mean response), the following can be said:

- If \(g\) is small, Bonferroni is better.
- If \(g\) is large, Working-Hotelling is better (the coefficient of \(s(\hat{Y}_h)\) in the confidence limits remains the same even as \(g\) becomes large).

Housing data as an example

Fitted regression model : \(\hat{Y}_h = 28.981 + 2.941X, n=19, s(b_0) = 8.5438, s(b_1) = 0.5412, MSE = 11.9512.\)

- **Simultaneous confidence intervals for \(\beta_0\) and \(\beta_1\)** : For 95% simultaneous C.I., \(t(1-\alpha/4; n-2)=t(0.0875;17) = 2.4581\). The intervals are (for \(\beta_0\) and \(\beta_1\), respectively)
Simultaneous inference for mean response at different values of $X$:

Say $g = 3$.

And the values are

<table>
<thead>
<tr>
<th>$X_h$</th>
<th>14</th>
<th>16</th>
<th>18.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\widehat{Y_h}$</td>
<td>70.155</td>
<td>76.037</td>
<td>83.390</td>
</tr>
<tr>
<td>$s(\widehat{Y_h})$</td>
<td>1.2225</td>
<td>0.8075</td>
<td>1.7011</td>
</tr>
</tbody>
</table>

$t(1-0.05/2g;n-2) = t(0.99167;17) = 2.655$, $\sqrt{2F(0.95;2,n-2)} = \sqrt{2 \times 3.5915} = 2.6801$

The 95% simultaneous confidence intervals for the mean responses are given in the following table:

<table>
<thead>
<tr>
<th>$X_h$</th>
<th>14</th>
<th>16</th>
<th>18.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonferroni</td>
<td>$70.155 \pm 3.248$</td>
<td>$76.037 \pm 2.145$</td>
<td>$83.390 \pm 4.520$</td>
</tr>
<tr>
<td>Working-Hotelling</td>
<td>$70.155 \pm 3.276$</td>
<td>$76.037 \pm 2.164$</td>
<td>$83.390 \pm 4.559$</td>
</tr>
</tbody>
</table>

Simultaneous prediction intervals for different values of $X$:

Again, say $g = 3$ and the values of 14, 16 and 18.5. In this case, $t(alpha = 0.05, t(1-alpha/2g;n-2) = t(0.99167; 17) = 2.655)$. And $\sqrt{gF(1-alpha;g,n-2)} = \sqrt{3F(0.95;3,17)} = \sqrt{3 \times 3.1968} = 3.0968)$. The standard errors and simultaneous 95% C.I. are given in the following table:

<table>
<thead>
<tr>
<th>$X_h$</th>
<th>14</th>
<th>16</th>
<th>18.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\widehat{Y_h}$</td>
<td>70.155</td>
<td>76.037</td>
<td>83.390</td>
</tr>
<tr>
<td>$s(Y_{h(new)}-\widehat{Y_h})$</td>
<td>3.6668</td>
<td>3.5501</td>
<td>3.8529</td>
</tr>
<tr>
<td>Bonferroni</td>
<td>$70.155 \pm 9.742$</td>
<td>$76.037 \pm 9.432$</td>
<td>$83.390 \pm 10.237$</td>
</tr>
<tr>
<td>Scheffe</td>
<td>$70.155 \pm 11.355$</td>
<td>$76.037 \pm 10.994$</td>
<td>$83.390 \pm 11.932$</td>
</tr>
</tbody>
</table>

Contributors

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