Simultaneous Inference

We want to find confidence interval for more than one parameter simultaneously. For example we might want to find confidence interval for **β_0** and **β_1**.

Bonferroni Joint Confidence Intervals

The confidence coefficients for individual parameters are adjusted to the higher 1 - α so that the confidence coefficient for the collection of parameters must be at least 1 - α. This is based on the following inequality:

Theorem (Bonferroni's Inequality)

$$P(\beta_0\cap\beta_1)\geq1-P(\beta_0^c)-P(\beta_1^c) \label{Bonferroni}$$

for any two events β_0 and β_1, where β_0^c and β_1^c are complements of events β_0 and β_1, respectively.

We take, $\beta_0 =$ the event that confidence interval for β_0 covers β_0; and, $\beta_1 =$ the event that confidence interval for **β_1** covers **β_1**;

So, if $P(\beta_0) = 1-\alpha_1$, and $P(\beta_1) = 1-\alpha_2$, then $P(\beta_0\cap\beta_1)\geq1-\alpha_1-\alpha_2$, by *Bonferroni's inequality* (Equation \ref{Bonferroni}). Note that $\beta_0\cap\beta_1$ is the event that confidence intervals for both the parameters cover the respective parameters. Therefore we take $\alpha_1 = \alpha_2 = \alpha/2$ to get joint confidence intervals with confidence coefficient at least $1 - \alpha$,

$b_0 \pm t(1-\alpha/4;n-2) s(b_0)$ *and* $b_1 \pm t(1-\alpha/4;n-2) s(b_1)$ for β_0 and β_1, respectively.

Bonferroni Joint Confidence Intervals for Mean Response

We want to find the simultaneous confidence interval for $E(Y|X = X_h) = \beta_0 + \beta_1X_h$ for *g* different values of X_h. Using *Bonferroni's inequality* for the intersection of *g* different events, the confidence intervals with confidence coefficient (at least) $1-\alpha$ are given by

$$\widehat{Y_h} \pm t(1-\alpha/2g; n-2)s(\widehat{Y_h}).$$

Confidence band for regression line : Working-Hotelling procedure

The confidence band

$$\widehat{Y_h} \pm\sqrt{2F(1-\alpha;2,n-2)}s(\widehat{Y_h})$$

contains the entire regression line (for all values of X) with confidence level $1-\alpha$. The Working-Hotelling procedure for obtaining the $1-\alpha$ simultaneous confidence band for the mean responses, therefore, is to use these confidence limits for the *g* different values of X_h.

Simultaneous prediction intervals

Recall that, the standard error of prediction for a new observation $Y_{h(new)}$ with $X = X_h$, is $$s(Y_{h(new)}-\widehat{Y_h}) = \sqrt{MSE(1+\frac{1}{n}+\frac{(X_h-\overline{X})^2}{\sum_i(X_i-\overline{X})^2})}$$ In order to predict the new observations for *g* different values of *X*, we may use one of the two procedures:

- **Bonferroni procedure :** $$\widehat{Y_h}\pm t(1-\alpha/2g;n-2)s(Y_{h(new)}-\widehat{Y_h})$$.
- **Scheffe procedure :** $$\widehat{Y_h}\pm \sqrt{gF(1-\alpha;g,n-2)}s(Y_{h(new)}-\widehat{Y_h})$$.

Remark : A point to note is that except for the Working-Hotelling procedure for finding simultaneous confidence intervals for mean response, in all the other cases, the confidence intervals become wider as *g* increases.

Which method to choose : Choose the method which leads to narrower intervals. As a comparison between Bonferroni and Working-Hotelling (for finding confidence intervals for the mean response), the following can be said :

- If *g* is small, Bonferroni is better.
- If *g* is large, Working-Hotelling is better (the coefficient of $s(\widehat{Y_h})$ in the confidence limits remains the same even as *g* becomes large).

Housing data as an example

Fitted regression model : $\widehat{Y_h} = 28.981 + 2.941X$, $n=19$, $s(b_0) = 8.5438$, $s(b_1) = 0.5412$, $MSE = 11.9512$.

- **Simultaneous confidence intervals for β_0 and β_1 :** For 95% simultaneous C.I., $t(1-\alpha/4;n-2)=t(0.9875;17) = 2.4581$. The intervals are (for β_0 and β_1, respectively)

\(28.981 \pm 2.4581 \times 8.5438 \equiv 28.981 \pm 21.002, 2.941 \pm 2.4581 \times 0.5412 \equiv 2.941 \pm 1.330\)

- **Simultaneous inference for mean response at *g* different values of X :** Say *g* = 3.

And the values are

$$X_h$$	14	16	18.5
$$\widehat{Y_h}$$	70.155	76.037	83.390
$$s(\widehat{Y_h})$$	1.2225	0.8075	1.7011

\(t(1-0.05/2g;n-2) = t(0.99167;17) = 2.655, \sqrt{2F(0.95;2,n-2)} = \sqrt{2 \times 3.5915} = 2.6801 \)

The 95% simultaneous confidence intervals for the mean responses are given in the following table:

$$X_h$$	14	16	18.5
Bonferroni	$$70.155 \pm 3.248$$	$$76.037 \pm 2.145$$	$$83.390 \pm 4.520$$
Working-Hotelling	$$70.155 \pm 3.276$$	$$76.037 \pm 2.164$$	$$83.390 \pm 4.559$$

- **Simultaneous prediction intervals for *g* different values of *X* :** Again, say *g* = 3 and the values of 14,16 and 18.5. In this case, \(\alpha = 0.05, t(1-\alpha/2g;n-2) = t(0.99167; 17) = 2.655\). And \(\sqrt{gF(1-\alpha;g,n-2)} = \sqrt{3F(0.95;3,17)} = \sqrt{3 \times 3.1968} = 3.0968\). The standard errors and simultaneous 95% C.I. are given in the following table:

$$X_h$$	14	16	18.5
$$\widehat{Y_h}$$	70.155	76.037	83.390
$$s(Y_{h(new)}-\widehat{Y_h})$$	3.6668	3.5501	3.8529
Bonferroni	$$70.155 \pm 9.742$$	$$76.037 \pm 9.432$$	$$83.390 \pm 10.237$$
Scheffe	$$70.155 \pm 11.355$$	$$76.037 \pm 10.994$$	$$83.390 \pm 11.932$$

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