Simultaneous Inference

We want to find confidence interval for more than one parameter simultaneously. For example we might want to find confidence interval for \( \beta_0 \) and \( \beta_1 \).

Bonferroni Joint Confidence Intervals

The confidence coefficients for individual parameters are adjusted to the higher 1 - \( \alpha \) so that the confidence coefficient for the collection of parameters must be at least 1 - \( \alpha \). This is based on the following inequality:

Theorem (Bonferroni's Inequality)

\[
P(\beta_0 \cap \beta_1) \geq 1 - P(\beta_0^c) - P(\beta_1^c) \tag{Bonferroni}\]

for any two events \( \beta_0 \) and \( \beta_1 \), where \( \beta_0^c \) and \( \beta_1^c \) are complements of events \( \beta_0 \) and \( \beta_1 \), respectively.

We take, \( \beta_0 = \) the event that confidence interval for \( \beta_0 \) covers \( \beta_0 \); and, \( \beta_1 = \) the event that confidence interval for \( \beta_1 \) covers \( \beta_1 \);

So, if \( P(\beta_0) = 1 - \alpha_1 \), and \( P(\beta_1) = 1 - \alpha_2 \), then \( P(\beta_0 \cap \beta_1) \geq 1 - \alpha_1 - \alpha_2 \), by Bonferroni’s inequality (Equation \ref{Bonferroni}). Note that \( \beta_0 \cap \beta_1 \) is the event that confidence intervals for both the parameters cover the respective parameters. Therefore we take \( \alpha_1 = \alpha_2 = \alpha/2 \) to get joint confidence intervals with confidence coefficient at least \( 1 - \alpha \),

\[
(b_0 \pm t(1-\alpha/4;n-2) s(b_0)) \text{ and } (b_1 \pm t(1-\alpha/4;n-2) s(b_1)) \text{ for } \beta_0 \text{ and } \beta_1\text{, respectively.}
\]
Bonferroni Joint Confidence Intervals for Mean Response

We want to find the simultaneous confidence interval for \(E(Y|X = X_h) = \beta_0 + \beta_1X_h\) for \(g\) different values of \(X_h\). Using Bonferroni’s inequality for the intersection of \(g\) different events, the confidence intervals with confidence coefficient (at least) \(1-\alpha\) are given by

\[
\widehat{Y_h} \pm t(1-\alpha/2g;n-2)s(\widehat{Y_h}).
\]

Confidence band for regression line: Working-Hotelling procedure

The confidence band

\[
\widehat{Y_h} \pm \sqrt{2F(1-\alpha;2,n-2)}s(\widehat{Y_h})
\]

contains the entire regression line (for all values of \(X_h\)) with confidence level \(1-\alpha\). The Working-Hotelling procedure for obtaining the \(1-\alpha\) simultaneous confidence band for the mean responses, therefore, is to use these confidence limits for the \(g\) different values of \(X_h\).

Simultaneous prediction intervals

Recall that, the standard error of prediction for a new observation \(Y_{(h(new))}\) with \(X = X_h\), is $s(Y_{(h(new))}-\widehat{Y_h}) = \sqrt{\text{MSE}(1+\frac{1}{n}+\frac{(X_h-\overline{X})^2}{\sum_i(X_i-\overline{X})^2})\text{sum}(X_i-\overline{X})^2)}$. In order to predict the new observations for \(g\) different values of \(X\), we may use one of the two procedures:

- **Bonferroni procedure**: $\widehat{Y_h} \pm t(1-\alpha/2g;n-2)s(Y_{h(new)}-\widehat{Y_h})$.
- **Scheffe procedure**: $\widehat{Y_h} \pm \sqrt{gF(1-\alpha;g,n-2)}s(Y_{(h(new))}-\widehat{Y_h})$.

**Remark**: A point to note is that except for the Working-Hotelling procedure for finding simultaneous confidence intervals for mean response, in all the other cases, the confidence intervals become wider as \(g\) increases.

**Which method to choose**: Choose the method which leads to narrower intervals. As a comparison between Bonferroni and Working-Hotelling (for finding confidence intervals for the mean response), the following can be said:

- If \(g\) is small, Bonferroni is better.
- If \(g\) is large, Working-Hotelling is better (the coefficient of \(s(\widehat{Y_h})\) in the confidence limits remains the same even as \(g\) becomes large).

Housing data as an example

Fitted regression model: \(\widehat{Y_h} = 28.981 + 2.941X, n=19, s(b_0) = 8.5438, s(b_1) = 0.5412, \text{MSE} = 11.9512.\)

- **Simultaneous confidence intervals for \(\beta_0\) and \(\beta_1\)**: For 95% simultaneous C.I., \(t(1-\alpha/4;n-2)=t(0.9875;17) = 2.4581\). The intervals are (for \(\beta_0\) and \(\beta_1\), respectively)
Simultaneous inference for mean response at \( g \) different values of \( X \): Say \( g = 3 \).

And the values are

<table>
<thead>
<tr>
<th>( X_h )</th>
<th>14</th>
<th>16</th>
<th>18.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{Y}_h )</td>
<td>70.155</td>
<td>76.037</td>
<td>83.390</td>
</tr>
<tr>
<td>( s(\hat{Y}_h) )</td>
<td>1.2225</td>
<td>0.8075</td>
<td>1.7011</td>
</tr>
</tbody>
</table>

\( t(1-0.05/2g;n-2) = t(0.99167;17) = 2.655, \sqrt{2F(0.95;2,n-2)} = \sqrt{2 \times 3.5915} = 2.6801 \)

The 95% simultaneous confidence intervals for the mean responses are given in the following table:

<table>
<thead>
<tr>
<th>( X_h )</th>
<th>14</th>
<th>16</th>
<th>18.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonferroni</td>
<td>$70.155 \pm 3.248$</td>
<td>$76.037 \pm 2.145$</td>
<td>$83.390 \pm 4.520$</td>
</tr>
<tr>
<td>Working-Hotelling</td>
<td>$70.155 \pm 3.276$</td>
<td>$76.037 \pm 2.164$</td>
<td>$83.390 \pm 4.559$</td>
</tr>
</tbody>
</table>

Simultaneous prediction intervals for \( g \) different values of \( X \): Again, say \( g = 3 \) and the values of 14, 16 and 18.5. In this case, \( \alpha = 0.05, t(1-\alpha/2g;n-2) = t(0.99167;17) = 2.655 \). And \( \sqrt{gF(1-\alpha;g,n-2)} = \sqrt{3F(0.95;3,17)} = \sqrt{3 \times 3.1968} = 3.0968 \). The standard errors and simultaneous 95% C.I. are given in the following table:

<table>
<thead>
<tr>
<th>( X_h )</th>
<th>14</th>
<th>16</th>
<th>18.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{Y}_h )</td>
<td>70.155</td>
<td>76.037</td>
<td>83.390</td>
</tr>
<tr>
<td>( s(\hat{Y}_{h(new)}-\hat{Y}_h) )</td>
<td>3.6668</td>
<td>3.5501</td>
<td>3.8529</td>
</tr>
<tr>
<td>Bonferroni</td>
<td>$70.155 \pm 9.742$</td>
<td>$76.037 \pm 9.432$</td>
<td>$83.390 \pm 10.237$</td>
</tr>
<tr>
<td>Scheffe</td>
<td>$70.155 \pm 11.355$</td>
<td>$76.037 \pm 10.994$</td>
<td>$83.390 \pm 11.932$</td>
</tr>
</tbody>
</table>

Contributors

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