Probability and Independence

For an experiment we define an event to be any collection of possible outcomes. A simple event is an event that consists of exactly one outcome.

- "or" means the union (i.e. either can occur)
- "and" means intersection (i.e. both must occur)

Two events are *mutually exclusive* if they cannot occur simultaneously. For a [Venn diagram](https://stats.libretexts.org/Bookshelves/Probability_Theory/Supplemental_Modules_(Probability)/Probability_and_Independence), we can tell that two events are mutually exclusive if their regions do not intersect.

Definition: Probability

We define *probability* of an event $\{E\}$ to be

\[
P(E) = \frac{\text{number of simple events within } E}{\text{ total number of possible outcomes}}\]

We have the following:

1. $P(E)$ is always between 0 and 1.
2. The sum of the probabilities of all simple events must be 1.
3. $P(E) + P(\text{not } E) = 1$
4. If $\{E\}$ and $\{F\}$ are mutually exclusive then

\[
P(E \text{ or } F) = P(E) + P(F)
\]
The Difference Between "and" and "or"

If \( \{E\} \) and \( \{F\} \) are events then we use the terminology

\[
E \text{ and } F
\]

to mean all outcomes that belong to both \( \{E\} \) and \( \{F\} \).

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Below is an example of two sets, \( \{A\} \) and \( \{B\} \), graphed in a Venn diagram.

The green area represents \( \{A\} \text{ and } \{B\} \) while all areas with color represent \( \{A\} \text{ or } \{B\} \).

Example

Our Women's Volleyball team is recruiting for new members. Suppose that a person inquires about the team.

- Let \( \{E\} \) be the event that the person is female
- Let \( \{F\} \) be the event that the person is a student

then \( \{E\} \text{ and } \{F\} \) represents the qualifications for being a member of the team. Note that \( \{E\} \text{ or } \{F\} \) is not enough.

We define

Definition: Conditional Probability

\[
P(E|F) = \frac{P(E \text{ and } F)}{P(F)}
\]

We read the left hand side as "The probability of event \( \{E\} \) given event \( \{F\} \) occurred."
We call two events *independent* if the following definitions hold.

**Definition: Independence**

For *independent* Events

\[ P(E|F) = P(E) \tag{1a} \]

Equivalently, we can say that \( E \) and \( F \) are independent if

**Definition: The Multiplication Rule**

For Independent Events

\[ P(E \text{ and } F) = P(E)P(F) \tag{1b} \]

**Example \( \PageIndex{2} \)**

Consider rolling two dice. Let

• \( E \) be the event that the first die is a 3.
• \( F \) be the event that the sum of the dice is an 8.

Then \( E \) and \( F \) means that we rolled a three and then we rolled a 5

This probability is 1/36 since there are 36 possible pairs and only one of them is (3,5)

We have

\[ P(E) = 1/6 \]

And note that (2,6),(3,5),(4,4),(5,3), and (6,2) give \( F \)

Hence

\[ P(F) = 5/36 \]

We have

\[ P(E)P(F) = (1/6)(5/36) \]

which is not 1/36 and we can conclude that \( E \) and \( F \) are not independent.

**Exercise \( \PageIndex{2} \)**

Test the following two events for independence:

• \( E \) the event that the first die is a 1.
• \( F \) the event that the sum is a 7.
A Counting Rule

For two events, \(E\) and \(F\), we always have

\[
P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) \label{2}
\]

Example \(\PageIndex{3}\)

Find the probability of selecting either a heart or a face card from a 52 card deck.

Solution

We let

- \(E\) = the event that a heart is selected
- \(F\) = the event that a face card is selected

then

\[
P(E) = \dfrac{1}{4}
\]

and

\[
P(F) = \dfrac{3}{13}
\]

that is, Jack, Queen, or King out of 13 different cards of one kind.

\[
P(E \text{ and } F) = \dfrac{3}{52}
\]

The counting rule formula (eq. 2) gives

\[
P(E \text{ or } F) = \dfrac{1}{4} + \dfrac{3}{13} - \dfrac{3}{52} = \dfrac{22}{52} = 42\text{%}
\]

Contributors

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