18.3: The Brownian Bridge

A Brownian bridge is a stochastic process $\{X_t : t \in [0, 1]\}$ with state space $\mathbb{R}$ that satisfies the following properties:

1. $X_0 = 0$ and $X_1 = 0$ (each with probability 1).
2. $\{\text{bs}(X)\}$ is a Gaussian process.
3. $\{E(X_t) = 0\}$ for $t \in [0, 1]$.
4. $\{\text{cov}(X_s, X_t) = \min\{s, t\} - st\}$ for $s, t \in [0, 1]$.
5. With probability 1, $t \mapsto X_t$ is continuous on $[0, 1]$.

Basic Theory

Definition and Constructions

In the most common formulation, the Brownian bridge process is obtained by taking a standard Brownian motion process $\{\text{bs}(X)\}$, restricted to the interval $[0, 1]$, and conditioning on the event that $X_1 = 0$. Since $X_0 = 0$ also, the process is "tied down" at both ends, and so the process in between forms a "bridge" (albeit a very jagged one).

The Brownian bridge turns out to be an interesting stochastic process with surprising applications, including a very important application to statistics. In terms of a definition, however, we will give a list of characterizing properties as we did for standard Brownian motion and for Brownian motion with drift and scaling.

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5. With probability 1, $t \mapsto X_t$ is continuous on $[0, 1]$. 
So, in short, a Brownian bridge \( \{X_t: t \in [0, 1]\} \) is a continuous Gaussian process with \( X_0 = X_1 = 0 \), and with mean and covariance functions given in (c) and (d), respectively. Naturally, the first question is whether there exists such a process. The answer is yes, of course, otherwise why would we be here? But in fact, we will see several ways of constructing a Brownian bridge from a standard Brownian motion. To help with the proofs, recall that a standard Brownian motion process \( \{Z_t: t \in [0, \infty)\} \) is a continuous Gaussian process with \( Z_0 = 0 \), \( \mathbb{E}(Z_t) = 0 \) for \( t \in [0, \infty) \) and \( \text{cov}(Z_s, Z_t) = \min\{s, t\} \) for \( s, t \in [0, \infty) \). Here is our first construction:

Suppose that \( \{Z_t: t \in [0, \infty)\} \) is a standard Brownian motion, and let \( X_t = Z_t - t Z_1 \) for \( t \in [0, 1] \). Then \( \{X_t: t \in [0, 1]\} \) is a Brownian bridge.

Proof

1. Note that \( X_0 = Z_0 = 0 \) and \( X_1 = Z_1 - Z_1 = 0 \).
2. Linear combinations of the variables in \( \{X_t: t \in [0, 1]\} \) reduce to linear combinations of the variables in \( \{Z_t: t \in [0, \infty)\} \) and hence have normal distributions. Thus \( \{X_t: t \in [0, 1]\} \) is a Gaussian process.
3. \( \mathbb{E}(X_t) = \mathbb{E}(Z_t) - t \mathbb{E}(Z_1) = 0 \) for \( t \in [0, 1] \).
4. \( \text{cov}(X_s, X_t) = \text{cov}(Z_s - s Z_1, Z_t - t Z_1) = \text{cov}(Z_s, Z_t) - t \text{cov}(Z_s, Z_1) - s \text{cov}(Z_1, Z_t) + s t \text{cov}(Z_1, Z_1) = \min\{s, t\} - s t + s t \) for \( s, t \in [0, 1] \).
5. \( t \mapsto X_t \) is continuous on \( [0, 1] \) since \( t \mapsto Z_t \) is continuous on \( [0, 1] \).

Let's see the Brownian bridge in action.

Run the simulation of the Brownian bridge process in single step mode a few times.

For the Brownian bridge \( \{X_t: t \in [0, 1]\} \), note in particular that \( X_t \) is normally distributed with mean 0 and variance \( t (1 - t) \) for \( t \in [0, 1] \). Thus, the variance increases and then decreases on \( [0, 1] \) reaching a maximum of \( 1/4 \) at \( t = 1/2 \). Of course, the variance is 0 at \( t = 0 \) and \( t = 1 \), since \( X_0 = X_1 = 0 \) deterministically.

Open the simulation of the Brownian bridge process. Vary \( t \) and note the change in the probability density function and moments. For various values of \( t \), run the simulation 1000 times and compare the empirical density function and moments to the true density function and moments.

Conversely to the construction above, we can build a standard Brownian motion on the time interval \( [0, 1] \) from a Brownian bridge.

Suppose that \( \{X_t: t \in [0, 1]\} \) is a Brownian bridge, and suppose that \( Z \) is a random variable with a standard normal distribution, independent of \( \{X_t: t \in [0, 1]\} \). Let \( Z_t = X_t + t Z \) for \( t \in [0, 1] \). Then \( \{Z_t: t \in [0, 1]\} \) is a standard Brownian motion on \( [0, 1] \).

Proof

1. Note that \( Z_0 = X_0 = 0 \).
2. Linear combinations of the variables in \( \{Z_t: t \in [0, 1]\} \) reduce to linear combinations of the variables in \( \{X_t: t \in [0, 1]\} \) and hence have normal distributions. Thus \( \{Z_t: t \in [0, 1]\} \) is a Gaussian process.
3. \( \mathbb{E}(Z_t) = \mathbb{E}(X_t) + t \mathbb{E}(Z) = 0 \) for \( t \in [0, 1] \).
4. \( \text{cov}(Z_s, Z_t) = \text{cov}(X_s + s Z, X_t + t Z) = \text{cov}(X_s, X_t) + s \text{cov}(X_s, Z) + s t \text{cov}(Z, Z) = \min\{s, t\} - s t + s t = \min\{s, t\} \) for \( s, t \in [0, 1] \).
5. \( t \mapsto Z_t \) is continuous on \( [0, 1] \) since \( t \mapsto X_t \) is continuous on \( [0, 1] \).
Here's another way to construct a Brownian bridge from a standard Brownian motion.

Suppose that \( \{\bs{Z}(t) : t \in [0, \infty)\} \) is a standard Brownian motion. Define \( \{X_1 = 0\} \) and \( \{X_t = (1 - t) Zt\}_{t \in [0, 1)} \) Then \( \{\bs{X}(t) : t \in [0, 1]\} \) is a Brownian bridge.

**Proof**

1. Note that \( X_0 = Z_0 = 0 \) and by definition, \( X_1 = 0 \).
2. Linear combinations of variables in \( \{\bs{X}(t) : t \in [0, 1]\} \) reduce to linear combinations of variables in \( \{\bs{Z}(t) : t \in [0, \infty)\} \) and hence have normal distributions. Thus \( \{\bs{X}(t) : t \in [0, 1]\} \) is a Gaussian process.
3. For \( t \in [0, 1] \), \( E[X_t] = (1 - t) E[Z_{\left(\frac{t}{1 - t}\right)}] = 0 \)
4. If \( s, t \in [0, 1] \) with \( s < t \) then \( \cov(X_s, X_t) = \cov(Z_{\left(\frac{s}{1 - s}\right)}, Z_{\left(\frac{t}{1 - t}\right)}) = (1 - s)(1 - t) \left(\frac{s}{1 - s} - \frac{s}{1 - s}\frac{t}{1 - t}\right) = s \)
5. Finally, \( \{t \mapsto X_t\} \) is continuous with probability 1 on \( [0, 1] \), and with probability 1, \( \{X_t = (1 - t) Z_{\left(\frac{t}{1 - t}\right)}\} \to 0 \) as \( t \to 0 \).

Conversely, we can construct a standard Brownian motion from a Brownian bridge.

Suppose that \( \{\bs{X}(t) : t \in [0, 1]\} \) is a Brownian bridge. Define \( \{Z_t = (1 + t) X_{\left(\frac{t}{1 + t}\right)}\}_{t \in [0, \infty)} \) Then \( \{\bs{Z}(t) : t \in [0, \infty)\} \) is a standard Brownian motion process.

**Proof**

1. Note that \( Z_0 = X_0 = 0 \)
2. Linear combinations of the variables in \( \{\bs{Z}(t) : t \in [0, \infty)\} \) reduce to linear combinations of the variables in \( \{X_t : t \in [0, 1]\} \), and hence have normal distributions. Thus \( \{\bs{Z}(t) : t \in [0, \infty)\} \) is a Gaussian process.
3. For \( t \in [0, \infty) \), \( E[Z_t] = (1 + t) E[X_{\left(\frac{t}{1 + t}\right)}] = 0 \)
4. If \( s, t \in [0, 1] \) with \( s < t \) then \( \cov(Z_s, Z_t) = \cov(X_{\left(\frac{s}{1 + s}\right)}, X_{\left(\frac{t}{1 + t}\right)}) = (1 + s)(1 + t) \left(\frac{s}{1 + s} - \frac{s}{1 + s}\frac{t}{1 + t}\right) = s \)
5. Since \( t \mapsto Z_t \) is continuous, \( t \mapsto \prod X_t \) is continuous

We return to the comments at the beginning of this section, on conditioning a standard Brownian motion to be 0 at time 1. Unlike the previous two constructions, note that we are not transforming the random variables, rather we are changing the underlying *probability measure*.

Suppose that \( \{\bs{X}(t) : t \in [0, \infty)\} \) is a standard Brownian motion. Then conditioned on \( \{X_1 = 0\} \), the process \( \{X_t : t \in [0, 1]\} \) is a Brownian bridge process.

**Proof**

Part of the argument is based on properties of the multivariate normal distribution. The conditioned process is still continuous and is still a Gaussian process. In particular, suppose that \( \{s, t \in [0, 1]\} \) with \( s < t \). Then \( \{X_t, X_1\} \) has a joint normal distribution with parameters specified by the mean and covariance functions of \( \{\bs{X}(t) \} \). By standard computations, the conditional distribution of \( \{X_t \} \) given \( \{X_1 = 0\} \) is normal with mean 0 and variance \( t(1 - t) \). Similarly, the joint distribution of \( \{X_s, X_t, X_1\} \) is normal with parameters specified by the mean and covariance functions of \( \{\bs{X}(t) \} \). Again, by standard computations, the conditional distribution of \( \{X_s, X_t\} \) given \( \{X_1 = 0\} \) is bivariate normal with 0 means and with \( \cov(X_s, X_t \mid X_1 = 0) = s(1 - t) \).
Finally, the Brownian bridge can be defined in terms a stochastic integral

Suppose that \( \{ \cdot | \bs{Z} = \{ Z_t : t \in [0, \infty) \} \} \) is standard Brownian motions. Define \( \{ X_{-1} = 1 \} \) and \( \{ X_1 = (1 - t) \int_0^t (1 - s) \, dZ_s, \quad t \in [0, 1] \} \) then \( \{ \bs{X} = \{ X_t : t \in [0, 1] \} \} \) is a Brownian bridge process.

Proof

1. Note that \( \{ X_0 = 0 \} \) and by definition, \( \{ X_1 = 0 \} \).
2. Since the integrand in the stochastic integral is deterministic, \( \{ \bs{X} \} \) is a Gaussian process.
3. \( \{ \bs{X} \} \) is continuous on \( [0, 1] \) with probability 1, as a basic property of stochastic integrals. Moreover, \( \{ X_t \to 0 \} \) as \( t \to 1 \) as a consequence of the martingale inequality.
4. \( \{ \E(X_t) = 0 \} \) since the stochastic integral has mean 0.
5. Suppose that \( s, t \in [0, 1] \) with \( s \leq t \). Then \( \{ \cov(X_s, X_t) = \cov\left[ (1 - s) \int_0^s \frac{1}{1 - u} \, dZ_u, \int_0^s \frac{1}{1 - u} \, dZ_u \right] \} \)
   \begin{align*}
   &\cov\left[ (1 - s) \int_0^s \frac{1}{1 - u} \, dZ_u, \int_s^t \frac{1}{1 - u} \, dZ_u \right] \\
   &\text{But then by the Ito isometry, } \{ \cov(X_s, X_t) = (1 - s)(1 - t) \int_0^s \frac{1}{(1 - u)^2} \, du \} \end{align*}
   \begin{align*}
   &= (1 - s)(1 - t) \int_0^s \frac{1}{(1 - u)^2} \, du = (1 - s)(1 - t) \left( \frac{1}{1 - s} - 1 \right) = (1 - t)s
   
In differential form, the process above can be written as \( dX_t = \frac{X_t}{1 - t} \, dt + dZ_t \), \( X_0 = 0 \).

The General Brownian Bridge

The processes constructed above (in several ways!) is the standard Brownian bridge. It's a simple matter to generalize the process so that it starts at \( \{ a \} \) and ends at \( \{ b \} \), for arbitrary \( \{ a, b \in \mathbb{R} \} \).

Suppose that \( \{ \bs{Z} = \{ Z_t : t \in [0, 1] \} \} \) is a standard Brownian process. Let \( \{ a, b \in \mathbb{R} \} \) and define \( \{ X_t = (1 - t) a + t b + Z_t \} \) for \( t \in [0, 1] \). Then \( \{ \bs{X} = \{ X_t : t \in [0, 1] \} \} \) is a Brownian bridge process from \( \{ a \} \) to \( \{ b \} \).

Of course, any of the constructions above for standard Brownian bridge can be modified to produce a general Brownian bridge. Here are the characterizing properties.

The Brownian bridge process \( \{ \bs{X} = \{ X_t : t \in [0, 1] \} \} \) from \( \{ a \} \) to \( \{ b \} \) is characterized by the following properties:

1. \( \{ X_0 = a \} \) and \( \{ X_1 = b \} \) (each with probability 1).
2. \( \{ \bs{X} \} \) is a Gaussian process.
3. \( \{ \E(X_t) = (1 - t) a + t b \} \) for \( t \in [0, 1] \).
4. \( \{ \cov(X_s, X_t) = \min\{s, t\} - s t \} \) for \( s, t \in [0, 1] \).
5. With probability 1, \( \{ t \mapsto X_t \} \) is continuous on \( [0, 1] \).

Applications
The Empirical Distribution Function

We start with a problem that is one of the most basic in statistics. Suppose that \( \{ T \} \) is a real-valued random variable with an unknown distribution. Let \( \{ F \} \) denote the distribution function of \( \{ T \} \), so that \( \{ F(t) = \mathbb{P}(T \leq t) \} \) for \( \{ t \in \mathbb{R} \} \). Our goal is to construct an estimator of \( \{ F \} \), so naturally our first step is to sample from the distribution of \( \{ T \} \). This generates a sequence \( \{ \text{bs}(T) = (T_1, T_2, \ldots) \} \) of independent variables, each with the distribution of \( \{ T \} \) (and so with distribution function \( \{ F \} \)). Think of \( \{ \text{bs}(T) \} \) as a sequence of independent copies of \( \{ T \} \). For \( \{ n \in \mathbb{N}_+ \} \) and \( \{ t \in \mathbb{R} \} \), the natural estimator of \( \{ F(t) \} \) based on the first \( \{ n \} \) sample values is \( \{ \text{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(T_i \leq t) \} \) which is simply the proportion of the first \( \{ n \} \) sample values that fall in the interval \( (-\infty, t] \). Appropriately enough, \( \{ \text{F}_n(t) \} \) is known as the empirical distribution function corresponding to the sample of size \( \{ n \} \). Note that \( \{ \text{left}[\{ \text{bs}(1) \} | T \leq t], \{ \text{bs}(1) \} | T \leq t], \{ \text{dots} \} \} \) is a sequence of independent, identically distributed indicator variables (and hence is a sequence of Bernoulli trials), and corresponds to sampling from the distribution of \( \{ \text{bs}(1) \} | T \leq t \). The estimator \( \{ \text{F}_n(t) \} \) is simply the sample mean of the first \( \{ n \} \) of these variables. The numerator, the number of the original sample variables with values in \( (-\infty, t] \), has the binomial distribution with parameters \( \{ n \} \) and \( \{ F(t) \} \). Like all sample means from independent, identically distributed samples, \( \{ \text{F}_n(t) \} \) satisfies some basic and important properties. A summary is given below, but to make sense of some of these facts, you need to recall the mean and variance of the indicator variable that we are sampling from: \( \{ \mathbb{E}[\{ \text{bs}(1) \} | T \leq t], \{ \var\{ \text{bs}(1) \} | T \leq t \} = F(t) \} \)

For fixed \( \{ t \in \mathbb{R} \} \),

1. \( \{ \mathbb{E}[\{ \text{F}_n(t) \}] = F(t) \} \) so \( \{ \text{F}_n(t) \} \) is an unbiased estimator of \( \{ F(t) \} \)
2. \( \{ \var\{ \text{F}_n(t) \} = F(t) | t \} \big/ \{ n \} \) so \( \{ \text{F}_n(t) \} \) is a consistent estimator of \( \{ F(t) \} \)
3. \( \{ \text{F}_n(t) \} \) is \( \{ n \} \)-to \( \{ \text{fmin} \} \) with probability 1, the strong law of large numbers.
4. \( \{ \text{sqr}(n) | \text{left}[\{ \text{F}_n(t) - F(t) \}] \} \) has mean 0 and variance \( \{ F(t) | t \} \) and converges to the normal distribution with these parameters as \( \{ n \} \)-to \( \{ \text{fmin} \} \), the central limit theorem.

The theorem above gives us a great deal of information about \( \{ \text{F}_n(t) \} \) for fixed \( \{ n \} \), but now we want to let \( \{ n \} \) vary and consider the expression in (d), namely \( \{ t \} \mapsto \text{sqr}(n) | \text{left}[\{ \text{F}_n(t) - F(t) \}] \), as a random process for each \( \{ n \} \in \{ \mathbb{N}_+ \} \). The key is to consider a very special distribution first.

Suppose that \( \{ T \} \) has the standard uniform distribution, that is, the continuous uniform distribution on the interval \( \{ 0, 1 \} \). In this case the distribution function is simply \( \{ F(t) = t \} \) for \( \{ t \} \in \{ 0, 1 \} \), so we have the sequence of stochastic processes \( \{ \text{bs}(X) \} \in \{ \text{left}[\{ X_n(t) : t \} | n \in \{ 0, 1 \}] \} \) for \( \{ n \} \in \{ \mathbb{N}_+ \} \), where \( \{ X_n(t) = \text{sqr}(n) | \text{left}[\{ F_n(t) - t \}] \} \). Of course, the previous results apply, so the process \( \{ \text{bs}(X) \} \) has mean function 0, variance function \( \{ t \} \mapsto t(1 - t) \), and for fixed \( \{ t \} \in \{ 0, 1 \} \), the distribution \( \{ X_n(t) \} \) converges to the corresponding normal distribution as \( \{ n \} \)-to \( \{ \text{fmin} \} \). Here is the new bit of information, the covariance function of \( \{ \text{bs}(X) \} \) is the same as that of the Brownian bridge!

\( \{ \text{cov}[\{ X_n(s), X_n(t) \}] = \text{min}\{s, t\} - s t \} \) for \( \{ s, t \} \in [0, 1] \).

Proof

Suppose that \( \{ s \} \in [0, 1] \). From basic properties of covariance, \( \{ \text{cov}[\{ X_n(s), X_n(t) \}] = n \} \cdot \{ \text{cov}[\{ F_n(s), F_n(t) \}] = \text{frac}(1) \} \cdot \{ \text{cov}[\{ \text{sum}_j | j = 1 \} \cdot \text{bs}(1) | T_i \leq s], \{ \text{sum}_j | j = 1 \} \cdot \text{bs}(1) | T_j \leq t \} \)
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov} \left[ \mathbb{1}(T_i \leq s) \mathbb{1}(T_j \leq t) \right] \] But if \( i \neq j \), the variables \( \mathbb{1}(T_i \leq s) \) and \( \mathbb{1}(T_j \leq t) \) are independent, and hence have covariance 0. On the other hand, \[
\text{cov} \left[ \mathbb{1}(T_i \leq s), \mathbb{1}(T_i \leq t) \right] = \mathbb{P}(T_i \leq s, T_i \leq t) - \mathbb{P}(T_i \leq s) \mathbb{P}(T_i \leq t) = \mathbb{P}(T_i \leq s) - \mathbb{P}(T_i \leq s) \mathbb{P}(T_i \leq t) = s - st \]
hence \[
\text{cov} \left[ X_n(s), X_n(t) \right] = \frac{1}{n} \sum_{i=1}^{n} \text{cov} \left[ \mathbb{1}(T_i \leq s), \mathbb{1}(T_i \leq t) \right] = s - st \]